Introductio

in Elementarem Functionum Ellipticarum Theoriam,

quam

Cons. Ampliss. Philos. Ord. Lundens.,

Præside

CAROLO JOH. Ds. HILL,

Math. Prof. Reg. & Ord.

Pro Laurea

p. p.

A. J. CLAESSON,

Blekingus.

In Acad. Carol. die XVIII Junii MDCCCXXXVIII.

P. XI.

LUNDÆ,
Typis excudit C. F. Berling,
Universitatis Typographus.
MDCCCXXXVIII.

$\mathfrak{S}^1\varphi = \sqrt{(\mathfrak{S}\varphi^2 - a^2).(\mathfrak{S}\varphi^2 - c^2)} = T\varphi . \dot{\mathfrak{S}}\varphi = \mathfrak{d}\mathfrak{S}\varphi$. Et tandem, quia

$\int_0^x \frac{dx}{X} = \iota B + \mathfrak{Q}x$, quando $X^2 = (x^2-b^2)(x^2+c^2) = x^4 + \overline{c^2-b^2}\,x^2 - \overline{bc}^2$

& igitur $\alpha = ibc = \frac{1}{\beta}$, erit similiter $\dot{\mathfrak{S}}(\varphi+\psi+\iota B) = \frac{\dot{\mathfrak{S}}\varphi.\dot{\mathfrak{S}}^1\psi + \dot{\mathfrak{S}}\psi.\dot{\mathfrak{S}}^1\varphi}{i(bc+\frac{1}{bc}.\dot{\mathfrak{S}}\varphi^2.\dot{\mathfrak{S}}\psi^2)}$.

Quæ utraque formula tantum sectoris imaginarii secantes explicat. Ipsæ vero novas sinuum formulas suppeditant. Quia enim $\iota\,\dot{\mathfrak{S}}(\varphi - \iota B) = \dot{S}\varphi$, erit $\dot{S}(\varphi+\psi) = \frac{\dot{\mathfrak{S}}\varphi.\dot{\mathfrak{S}}^1\psi + \dot{\mathfrak{S}}\psi.\dot{\mathfrak{S}}^1\varphi}{bc+\frac{1}{bc}.\dot{\mathfrak{S}}\varphi^2.\dot{\mathfrak{S}}\psi^2}$. In formula vero ipsius $\mathfrak{S}$ primum φ & ψ negative sumantur, ut evadat

$\mathfrak{S}(A+\iota B-\varphi-\psi) = -\frac{\mathfrak{S}\varphi.\mathfrak{S}^1\psi+\mathfrak{S}\psi.\mathfrak{S}^1\varphi}{ac-\frac{1}{ac}.\mathfrak{S}\varphi^2.\mathfrak{S}\psi^2}$, & deinde, quia

$\mathfrak{S}\psi = S(\psi+A+\iota B)$, erit $S\psi = \mathfrak{S}(\psi - A - \iota B)$ seu $-S(\psi+\varphi) = \mathfrak{S}(A+\iota B+\psi+\varphi)$ *), ideoque $S\overline{\varphi+\psi} = \frac{\mathfrak{S}\varphi.\mathfrak{S}^1\psi+\mathfrak{S}\psi.\mathfrak{S}^1\varphi}{-ac+\frac{1}{ac}.\mathfrak{S}\varphi^2.\mathfrak{S}\psi^2}$. Sin vero ponimus $-\varphi$ pro φ, ponendum est $-\mathfrak{S}^1\varphi$ pro $\mathfrak{S}^1\varphi$, ut sit $S(\psi-\varphi) = ac.\frac{\mathfrak{S}\varphi.\mathfrak{S}^1\psi - \mathfrak{S}\psi.\mathfrak{S}^1\varphi}{\mathfrak{S}\varphi^2.\mathfrak{S}\psi^2-(ac)^2}$; quia $\mathfrak{S}^1\varphi = \dot{\mathfrak{S}}\varphi.T\varphi$. Ita igitur id, quod quærebamus, nempe $\mathfrak{T}\overline{\varphi+\psi}$, $\mathfrak{S}\overline{\varphi+\psi}$, $\dot{\mathfrak{S}}\overline{\varphi+\psi}$, explicitas, non invenire videmur, sed novas tantum formulas, quæ ad functiones

---

*) Hic valor etiam ex formula generali elicitur, ponendo $\varphi = o$, ideoque $\mathfrak{d}\mathfrak{S}\varphi = o$, $\mathfrak{S}o = a$. Ita enim fit: $\mathfrak{S}(A+\iota B+\psi) = \frac{\mathfrak{S}o.\mathfrak{S}^1\psi.ac}{(ac)^2-\mathfrak{S}o^2.\mathfrak{S}\psi^2} = \frac{c.T\psi\,\dot{\mathfrak{S}}\psi}{c^2-\mathfrak{S}\psi^2} = \frac{-cT\psi}{\dot{\mathfrak{S}}\psi} = -S\psi$.

$T(\varphi+\psi)$, $S(\varphi+\psi)$ atque $\dot{S}(\varphi+\psi)$ explicandas seu cum functionibus sectorum φ & ψ comparandas inserviant. Nihilo tamen minus ejusmodi explicatio reperiatur, quia (p. 20) $\mathfrak{T}(\varphi+\psi)=a.\sqrt{\frac{b^2+T(\varphi+\psi)^2}{a^2+T(\varphi+\psi)^2}}$, $\dot{\mathfrak{S}}(\varphi+\psi)=\sqrt{b^2+T(\varphi+\psi)^2}$. & $T\overline{\varphi+\psi}$ jam explicita est. Sin vero hujus valorem substitueremus, tiro, radicem revera extrahi posse, non facile animadverteret, nec hoc absque ambagibus perficeret. Videamus igitur, num id, quod quærimus, ex ipsis formulis jam inventis extorquere liceat. Quem in finem ponamus $\psi+iB=\chi$, eritque

$$i\mathfrak{T}\overline{\varphi+\chi}=i\mathfrak{T}(\varphi+\psi+iB)=\frac{\mathfrak{T}\varphi.\mathfrak{T}'(\chi-iB)+\mathfrak{T}(\chi-iB).\mathfrak{T}'\varphi}{ab-\frac{1}{ab}.\mathfrak{T}\varphi^2.\mathfrak{T}(\chi-iB)^2}.$$

Est vero $G1)$ (p 48) $\mathfrak{T}\psi=\mathfrak{T}(\chi-iB)=\pm i.T\chi$, & igitur $\mathfrak{T}'(\chi-iB)=\mathfrak{T}'\psi=i.T'\chi$, ideoque $\mathfrak{T}(\varphi+\chi)=\frac{\mathfrak{T}\varphi\,.\,T'\chi+T\chi\,.\mathfrak{T}'\varphi}{ab+\frac{1}{ab}.\mathfrak{T}\varphi^2.T\chi^2}$.

Similiter secantium transformantur formulæ. Quia enim ($G6$) $\dot{\mathfrak{S}}\psi=\dot{\mathfrak{S}}(\chi-iB)=i\dot{S}\chi$, ideoque $\dot{\mathfrak{S}}'\psi=i\dot{S}'\chi$, erit

$$\dot{\mathfrak{S}}(\varphi+\chi)=\frac{\dot{\mathfrak{S}}\varphi.\dot{S}'\chi+\dot{S}\chi.\dot{\mathfrak{S}}'\varphi}{bc-\frac{1}{bc}.\dot{\mathfrak{S}}\varphi^2.\dot{S}\chi^2},$$ existente $\dot{S}'\chi=S^c\chi.\mathfrak{T}\chi$ & $\dot{\mathfrak{S}}'\varphi=\mathfrak{S}\varphi.T\varphi$.

Posito deinde $\psi+A+iB=\eta$, erit $\mathfrak{S}\overline{\varphi+\eta}=\frac{\mathfrak{S}\varphi.\mathfrak{S}'\psi+\mathfrak{S}\psi\,\mathfrak{S}'\varphi}{ac-\frac{1}{ac}.\mathfrak{S}\varphi^2.\mathfrak{S}\psi^2}$.

Est vero ($G5$) $\mathfrak{S}\psi=\mathfrak{S}(\eta-A-iB)=S\eta$, ideoque $\mathfrak{S}'\psi=S'\eta$, atque tandem $\mathfrak{S}(\varphi+\eta)=\frac{\mathfrak{S}\varphi.S'\eta+S\eta.\mathfrak{S}'\varphi}{ac-\frac{1}{ac}.\mathfrak{S}\varphi^2\,S\eta^2}$; (ubi $S'\eta=\dot{S}^c\eta\,.\mathfrak{T}^c\eta$ & $\mathfrak{S}'\varphi=\dot{\mathfrak{S}}\varphi.T\varphi$). Quoniam vero tironibus via imaginaria, qua has formulas jam invenimus, dubiosa videri possit, in ipsas & aliter inquirere lubet.

Formulas scilicet habemus secantes exprimentes reciprocas, nempe $\mathfrak{S}\varphi=\frac{ac}{S^c\varphi}$, atque $\dot{\mathfrak{S}}\varphi=\frac{bc}{\dot{S}^c\varphi}$; quæramus igitur utrumque cosinum

summæ, id quod facile efficitur sumendo $A-\psi$ seu ${}^c\psi$ pro ψ in formulis differentiæ. Est enim $F^c(\varphi+\psi) = F(A-\psi-\varphi) = F({}^c\psi-\varphi)$;

ideoque $S^c(\varphi+\psi) = S({}^c\psi-\varphi) = \dfrac{S({}^c\psi) . S'\varphi - S\varphi . S'({}^c\psi)}{S'o - \frac{1}{S'o} . S\varphi^2 . \widehat{S^c\psi^2}}$, seu

$$S^c(\varphi+\psi) = \frac{S^c\psi . S'\varphi + S\varphi . \overset{c\,1}{S\psi}}{S'o-(S'o)^{-1}.(S\varphi)^2.(S^c\psi)^2},$$ existente $S'o = ac$, $S'\varphi = \dot{S}^c\varphi.\mathfrak{T}^c\varphi$,

ideoque $S'({}^c\psi) = \dot{S}^c({}^c\psi) . \mathfrak{T}\psi = \dot{S}\psi . \mathfrak{T}\psi = -\overset{1}{S^c\psi}$. Sin vero alteram formulam, modo nempe inventam, adhibuerimus, erit

$$S^c(\varphi+\psi) = S({}^c\psi-\varphi) = ac . \frac{\mathfrak{S}\varphi . \mathfrak{S}'({}^c\psi) - \mathfrak{S}^c\psi . \mathfrak{S}'\varphi}{\mathfrak{S}\varphi^2 . \widehat{\mathfrak{S}^c\psi^2} - \widehat{ac^2}},$$ ubi

$\mathfrak{S}'\varphi = \dot{\mathfrak{S}}\varphi . T\varphi$, ideoque $\mathfrak{S}'({}^c\psi) = \dot{\mathfrak{S}}^c\psi . T^c\psi$. Erit igitur

$$\mathfrak{S}(\varphi+\psi) = \frac{ac}{S^c(\varphi+\psi)} = \frac{\widehat{ac^2} - S\varphi^2 . S^c\psi^2}{S^c\psi . \dot{S}^c\varphi\,\mathfrak{T}^c\varphi - S\varphi . \dot{S}\psi\,\mathfrak{T}\psi},$$ itemque

$$\widehat{\mathfrak{S}\varphi+\psi} = \frac{\mathfrak{S}\varphi^2 . \widehat{\mathfrak{S}^c\psi^2} - \widehat{ac^2}}{\mathfrak{S}\varphi . \mathfrak{d}\,\mathfrak{S}^c\psi - \mathfrak{S}^c\psi . \mathfrak{d}\,\mathfrak{S}\varphi} = \frac{\mathfrak{S}\varphi^2 . \widehat{\mathfrak{S}^c\psi^2} - \widehat{ac^2}}{\mathfrak{S}\varphi . \dot{\mathfrak{S}}^c\psi . T^c\psi - \mathfrak{S}^c\psi . \dot{\mathfrak{S}}\varphi . T\varphi}.$$

Similiter inveniemus $\dot{S}^c(\varphi+\psi)$ atque $\dot{\mathfrak{S}}(\varphi+\psi)$. Est enim

$$\dot{S}^c(\psi+\varphi) = \dot{S}({}^c\psi-\varphi) = \frac{\dot{S}^c\psi . \dot{S}'\varphi - \dot{S}\varphi . \dot{S}'({}^c\psi)}{bc + \frac{1}{bc} . \dot{S}\varphi^2 . \widehat{\dot{S}^c\psi^2}},$$

ubi $\dot{S}'\varphi = S^c\varphi . \mathfrak{T}\varphi$, & $\dot{S}'({}^c\psi) = S\psi . \mathfrak{T}^c\psi$, ideoque

$$\dot{\mathfrak{S}}(\varphi+\psi) = \frac{bc}{\dot{S}^c(\varphi+\psi)} = \frac{\widehat{bc^2} + \dot{S}\varphi^2 . \widehat{\dot{S}^c\psi^2}}{\dot{S}^c\psi\, S^c\varphi . \mathfrak{T}\varphi - \dot{S}\varphi . S\psi\,\mathfrak{T}^c\psi}.$$ Vel etiam,

quia $\dfrac{\dot{S}(\varphi+\psi)}{bc} = \dfrac{\dot{\mathfrak{S}}\varphi . \mathfrak{S}\psi . T\psi + \dot{\mathfrak{S}}\psi . \mathfrak{S}\varphi\, T\varphi}{\widehat{bc^2} + \dot{\mathfrak{S}}\varphi^2 . \dot{\mathfrak{S}}\psi^2}$, ideoque

$$\frac{\dot{S}(\psi-\varphi)}{bc} = \frac{\dot{\mathfrak{S}}\varphi . \mathfrak{S}\psi . T\psi - \dot{\mathfrak{S}}\psi\,\mathfrak{S}\varphi\, T\varphi}{\widehat{bc^2} + \dot{\mathfrak{S}}\varphi^2 . \dot{\mathfrak{S}}\psi^2},$$

atque $\frac{\dot{S}^c(\psi+\varphi)}{bc} = \frac{\dot{S}({}^c\psi-\varphi)}{bc} = \frac{\mathfrak{S}\varphi.\mathfrak{S}^c\psi T^c\psi - \dot{\mathfrak{S}}^c\psi.\mathfrak{S}\varphi\, T\varphi}{\widehat{bc}^2 + \dot{\mathfrak{S}}\varphi^2 . \dot{\mathfrak{S}}^c\widehat{\psi}^2}$,

erit $\dot{\mathfrak{S}}\overline{\varphi+\psi} = \frac{\widehat{bc}^2 + \dot{\mathfrak{S}}\varphi^2 . \dot{\mathfrak{S}}^c\widehat{\varphi}^2}{\dot{\mathfrak{S}}\varphi.\mathfrak{S}^c\psi\, T^c\psi - \dot{\mathfrak{S}}^c\psi.\mathfrak{S}\varphi\, T\varphi}$, seu (permutatis φ & ψ)

$\dot{\mathfrak{S}}(\varphi+\psi) = \frac{\widehat{bc}^2 + \dot{\mathfrak{S}}\psi^2 . \dot{\mathfrak{S}}^c\widehat{\varphi}^2}{\dot{\mathfrak{S}}\psi.\mathfrak{S}^c\varphi\, T^c\varphi - \dot{\mathfrak{S}}^c\varphi.\mathfrak{S}\psi\, T\psi}$. Hæ formulæ ab illis antea inventis, quæ secantes summæ explicabant, forma quidem differunt, revera tamen reductionibus ex pág. 20 petitis, per aliquas ambagines convenire videbuntur. Tandem cosecantes invenientur, vel ex his secundum formulam $\mathfrak{S}^c(\varphi+\psi) = \mathfrak{S}(A-\psi-\varphi) = \mathfrak{S}({}^c\psi-\varphi)$, vel secundum has: $\mathfrak{S}^c\overline{\varphi+\psi} = \frac{ac}{S(\varphi+\psi)}$, & $\dot{\mathfrak{S}}^c(\varphi+\psi) = \frac{bc}{\dot{S}\overline{\varphi+\psi}}$.

Exhinc igitur erit $\mathfrak{S}^c(\varphi+\psi) = \frac{\mathfrak{S}\varphi^2\,\mathfrak{S}\psi^2 - \widehat{ac}^2}{\mathfrak{S}\varphi.\mathfrak{S}'\psi + \mathfrak{S}\psi.\mathfrak{S}'\varphi}$, atque

$\dot{\mathfrak{S}}^c(\varphi+\psi) = \frac{\dot{\mathfrak{S}}\varphi^2 . \dot{\mathfrak{S}}\psi^2 + (bc)^2}{\dot{\mathfrak{S}}\varphi\,\dot{\mathfrak{S}}'\psi + \dot{\mathfrak{S}}\psi.\dot{\mathfrak{S}}'\varphi}$; sin vero alteras adhibuerimus formulas, erit $\mathfrak{S}^c(\varphi+\psi) = \frac{\widehat{ac}^2 - S\varphi^2 . S\psi^2}{S\varphi\,\mathfrak{d}\,S\psi + S\psi\,\mathfrak{d}\,S\varphi}$, atque

$\dot{\mathfrak{S}}^c(\varphi+\psi) = \frac{(bc)^2 + \dot{S}\varphi^2 . \widehat{\dot{S}\psi}^2}{\dot{S}\varphi\,\mathfrak{d}\,\dot{S}\psi + \dot{S}\psi\,\mathfrak{d}\,\dot{S}\varphi} = \frac{(bc)^2 + \dot{S}\varphi^2 . \dot{S}\psi^2}{\dot{S}\varphi . S^c\psi\,\mathfrak{T}\psi + \dot{S}\psi . S^c\varphi . \mathfrak{T}\varphi}$.

Restant igitur tantummodo valores explicandi functionum $\mathfrak{T}(\varphi+\psi)$ & $\mathfrak{T}^c(\varphi+\psi)$. Quia vero $\mathfrak{T}\varphi = a\frac{\dot{S}\varphi}{S\varphi}$, erit

$\mathfrak{T}(\varphi+\psi) = a . \frac{\dot{S}\varphi\,\mathfrak{d}\,\dot{S}\psi + \dot{S}\psi\,\mathfrak{d}\,\dot{S}\varphi}{bc + \frac{1}{bc} . \dot{S}\varphi^2 . \dot{S}\psi^2} . \frac{ac - \frac{1}{ac} . S\varphi^2 . S\psi^2}{S\varphi\,\mathfrak{d}\,S\psi + S\psi\,\mathfrak{d}\,S\varphi}$, ubi

$\mathfrak{d}\,\dot{S}\varphi = S^c\varphi . \mathfrak{T}\varphi$, atque $\mathfrak{d}S\varphi = \dot{S}^c\varphi . \mathfrak{T}^c\varphi$; ideoque ita habetur valor ipsius

$\mathfrak{T}(\varphi+\psi)$ sub forma proposita. Hæc vero formula satis complicatam præ se fert speciem, nec in promtu est videre, quarum reductionum cujusque divisoris ope simplicior efficiatur. Nullum vero est dubium, quin algebraice in formam modo inventam transmutari possit. Hujusmodi vero ambages evitaturi, aliam hujus rei comprobandi caussa viam ingrediamur, & expressionem adhibeamus ipsius $\mathfrak{T}$ per T imaginariam: $\mathfrak{T}\varphi=\frac{1}{i}T(\varphi+iB)$; quam tamen, ut tutius procedatur, prævie per ipsam tangentis formulam comprobamus. Ponamus igitur $T\psi=ib$, eritque $\mathfrak{S}\psi=o=\mathfrak{d}T\psi$, ideoque

$$T(\psi+\varphi)=ab\,.\,\frac{T\varphi\,\mathfrak{d}\,T\psi+T\psi\,\mathfrak{d}\,T\varphi}{(ab)^2-T\varphi^2\,.\,T\psi^2}=ab\,.\,\frac{ib\,.\,\mathfrak{d}\,T\varphi}{a^2\,b^2+b^2\,.\,T\varphi^2},$$

seu $$T(\varphi+\psi)=\frac{i\,a\,.\,\mathfrak{d}\,T\varphi}{a^2+T\varphi^2}=i\,.\,\frac{a\sqrt{a^2+T\varphi^2}\,.\,\sqrt{b^2+T\varphi^2}}{a^2+T\varphi^2}=$$

$$=i\,a\,.\sqrt{\frac{b^2+T\varphi^2}{a^2+T\varphi^2}},$$ ideoque $T(\varphi+\psi)=i\,.\,\mathfrak{T}\varphi$, siquidem fuerit $T\psi=ib$, seu $\psi=\mathfrak{L}\,\widehat{ib}$, quod jam antea ipso $i\,B=i\,.\,\mathfrak{L}\,\overline{\frac{a}{c},\infty}=i\,\mathfrak{L}'\infty$ (exsistente $a'=a$, & $b'=c$) æquare docuimus. *) — Quoniam igitur certe $T(\varphi+iB)=i\,\mathfrak{T}\varphi$ sit, ponamus $\varphi+iB$ pro φ' in formula

$$T(\varphi'+\psi)=\frac{T\psi\,\mathfrak{d}\,T\varphi'+T\varphi'\,\mathfrak{d}\,T\psi}{ab-\frac{1}{ab}\,.\,T\varphi'^2\,.\,T\psi^2},$$ eritque, quia $T\varphi'=T(\varphi+iB)=i\,\mathfrak{T}\varphi$, ideoque $\widehat{T\varphi'^2}=-\mathfrak{T}\varphi^2$, tum

$$\mathfrak{d}\,T\varphi'=\sqrt{a^2+T\varphi'^2}\,.\,\sqrt{b^2+T\varphi'^2}=\sqrt{a^2-\mathfrak{T}\varphi^2}\,.\,\sqrt{b^2-\mathfrak{T}\varphi^2}=\pm\,i\,\mathfrak{d}\mathfrak{T}\varphi,$$

tum $$i\mathfrak{T}(\varphi+\psi)=T(\psi+\varphi+iB)=\frac{\pm\,T\psi\,.\,i\,\mathfrak{d}\mathfrak{T}\varphi+i\,\mathfrak{T}\varphi\,.\,\mathfrak{d}\,T\psi}{ab+\frac{1}{ab}\,.\,\mathfrak{T}\varphi^2\,.\,T\psi^2},$$ seu

$$\mathfrak{T}(\varphi+\psi)=\frac{\mathfrak{T}\varphi\,.\,\mathfrak{d}\,T\psi+T\psi\,.\,\mathfrak{d}\mathfrak{T}\varphi}{ab+\frac{1}{ab}\,.\,\mathfrak{T}\varphi^2\,.\,T\psi^2},$$ quæ formula illa est simplicissima

---

*) Si posueris $T\psi=ia$, similiter invenies $T(\varphi+\mathfrak{L}\,i\,a)=i\,\mathfrak{T}^c\varphi$.

(eoque ipso elegantissima) subtangentis, & novum hujus cum tangente nexum exhibentis, quam antea aliter inveniebamus. Hinc vero immediate elicitur $\left(\frac{ab}{\mathfrak{T}\overline{\varphi + \psi}} = \right) \mathfrak{T}^c(\varphi + \psi) = \frac{(ab)^2 + \mathfrak{T}\varphi^2 \,.\, T\psi^2}{\mathfrak{T}\varphi \,.\, \mathfrak{d}\,T\psi + T\psi\,\mathfrak{d}\,\mathfrak{T}\varphi}$; quod & aliter exprimi potest. Nam $\mathfrak{T}^c(\varphi + \psi) = \mathfrak{T}({}^c\varphi - \psi) =$

$$= ab \,.\, \frac{\mathfrak{T}^c\varphi\,\mathfrak{d}\,T\psi - T\psi\,\mathfrak{d}\,\mathfrak{T}({}^c\varphi)}{(ab)^2 + \mathfrak{T}^c\varphi^2 \,.\, T\psi^2}, \text{ seu } = \mathfrak{T}({}^c\psi - \varphi) =$$

$$= ab \,.\, \frac{\mathfrak{T}\varphi\,\mathfrak{d}\,T({}^c\psi) - T^c\psi \,.\, \mathfrak{d}\mathfrak{T}\varphi}{(ab)^2 + \mathfrak{T}\varphi^2 \,.\, T^c\psi^2},$$ existente $\mathfrak{d}T({}^c\psi) = \mathfrak{S}^c\psi \,.\, \dot{\mathfrak{S}}^c\psi$, &

$\mathfrak{d}\,\mathfrak{T}({}^c\varphi) = S\varphi \,.\, \dot{S}^c\varphi$, atque $\mathfrak{d}\,\mathfrak{T}\varphi = S^c\varphi . S\dot{\varphi}$. Prima harum formularum facile mutatur in $\mathfrak{T}^c(\varphi + \psi) = ab \,.\, \frac{\mathfrak{T}^c\varphi^2 + T\psi^2}{\mathfrak{T}^c\varphi \,.\, \mathfrak{d}\,T\psi - T\psi \,.\, \mathfrak{d}\,\mathfrak{T}^c\varphi}$; ex qua mox elicitur: $\mathfrak{T}^c(\varphi + \psi) = ab \,.\, \frac{\mathfrak{T}^c\varphi\,\mathfrak{d}\,T\psi + T\psi\,\mathfrak{d}\,\mathfrak{T}^c\varphi}{\widehat{ab}^2 + T\psi^2 \,.\, \mathfrak{T}^c\varphi^2}$, et deinde colligitur:

$$\mathfrak{T}^c(\varphi + \psi) = \frac{\mathfrak{T}\varphi \,.\, \mathfrak{S}\psi \,.\, \dot{\mathfrak{S}}\psi - T\psi\,\dot{S}\varphi\,S^c\varphi}{\mathfrak{T}\varphi^2 + T\psi^2},$$ (in usum vocando nempe formulas elementares $ab = \mathfrak{T}\,\mathfrak{T}^c$, $\mathfrak{T} \,.\, S = a\dot{S}$ & $\mathfrak{T} \,.\, \dot{S}^c = bS^c$),

seu $\mathfrak{T}^c(\varphi + \psi) = \frac{\mathfrak{T}\varphi\,\mathfrak{d}\,T\psi - T\psi\,\mathfrak{d}\,\mathfrak{T}\varphi}{\mathfrak{T}\varphi^2 + T\psi^2}$. Videmus igitur, formulam addendæ functionis hujus variis modis exprimi posse; id quod ad qvamcunque reliquarum facile reperies. Sic ex gr. invenitur iisdem modis: Ex formula tangentis

$$\frac{T(\varphi + \psi)}{ab} = \frac{T\varphi\,\mathfrak{d}\,T\psi + T\psi\,\mathfrak{d}\,T\varphi}{(ab)^2 - T\varphi^2\,T\psi^2},$$ (primum, si lubet, ponendo $\psi = -A$, & $T(.A) = \infty$, ut inveniatur $T(A - \varphi)$ seu $T^c\varphi = \frac{ab}{T\varphi}$, formulæ elementari convenienter),

$T^c(\varphi + \psi) = \frac{(ab)^2 - T\varphi^2 . T\psi^2}{T\varphi\, \mathfrak{d}T\psi + T\psi\, \mathfrak{d}T\varphi}$, & $= \frac{T^c\varphi\, \mathfrak{d}T\psi - T\psi\, \mathfrak{d}T(^c\varphi)}{(ab)^2 - T^c\varphi^2 . T\psi^2} . ab$, seu

$T^c(\varphi + \psi) = ab . \frac{T^c\varphi . \mathfrak{S}\psi . \dot{\mathfrak{S}}\psi - T\psi\, \mathfrak{S}^c\varphi . \dot{\mathfrak{S}}^c\varphi}{(ab)^2 - \widehat{T^c\varphi}^2 . T\psi^2}$, & hinc, (ob $ab = T . T^c$, $a\dot{\mathfrak{S}} = T . \mathfrak{S}^c$ atque $b\mathfrak{S} = T . \dot{\mathfrak{S}}^c$) (v. p. 35)

$T^c(\varphi + \psi) = \frac{T^c\varphi . \dot{\mathfrak{S}}^c\psi . \mathfrak{S}^c\psi - T^c\psi . \mathfrak{S}^c\varphi . \dot{\mathfrak{S}}^c\varphi}{T^c\psi^2 - T^c\varphi^2}$, seu

$T^c(\varphi + \psi) = \frac{T^c\psi . \mathfrak{d}T^c\varphi - T^c\varphi . \mathfrak{d}T^c\psi}{\widehat{T^c\psi}^2 - \widehat{T^c\varphi}^2}$. Similiter ex inventis cosecantium formulis obtinemus, (quia $\mathfrak{d}S = \dot{S}^c . \mathfrak{T}^c$, $ac = S . \mathfrak{S}^c$, $a . \dot{S}^c = T^c . S$, & $c . \mathfrak{T}^c = \dot{\mathfrak{S}}^c . S$), $\mathfrak{S}^c(\varphi + \psi) = \frac{\mathfrak{S}^c\varphi^2 . \mathfrak{S}^c\psi^2 - (ac)^2}{\mathfrak{S}^c\varphi . T^c\psi . \dot{\mathfrak{S}}^c\psi + \mathfrak{S}^c\psi . T^c\varphi . \dot{\mathfrak{S}}^c\varphi}$; nec non

$\mathfrak{S}^c(\varphi + \psi) = a . \frac{\mathfrak{S}^c\varphi . \mathfrak{S}^c\psi - S\varphi . S\psi}{T^c\psi . \mathfrak{T}^c\psi + T^c\varphi . \mathfrak{T}^c\varphi}$: Itemque (quia $\mathfrak{d}\dot{S} = S^c . \mathfrak{T}$, $bc = \dot{S}\dot{\mathfrak{S}}^c$, atque $bS^c = \dot{S}\, T^c$)

$\dot{\mathfrak{S}}^c(\varphi + \psi) = b . \frac{\dot{\mathfrak{S}}^c\varphi . \dot{\mathfrak{S}}^c\psi + \dot{S}\varphi . \dot{S}\psi}{\mathfrak{T}\psi . T^c\psi + \mathfrak{T}\varphi . T^c\varphi}$, atque $= \frac{\dot{\mathfrak{S}}^c\varphi^2 . \dot{\mathfrak{S}}^c\psi^2 + b^2c^2}{\dot{\mathfrak{S}}^c\varphi . T^c\psi . \mathfrak{S}^c\psi + \dot{\mathfrak{S}}^c\psi . T^c\varphi . \mathfrak{S}^c\varphi}$.

Exhinc vero novas facile elicies formulas sinuum, sicuti tangentium ex cotangentibus. Quamplurimis igitur modis formulæ summæ et differentiæ exprimi possunt; omnium vero ii præcipue eligendi sunt, qui partim functionis summæ per ipsas functiones, de quarum summis inversis quæritur, absque fractione fractionum exprimunt, & partim ii, qui minimum utrinque factorum numerum admittunt. Ad primum itaque modum referendæ sunt eæ formulæ, quæ $F(\varphi \pm \psi)$ per $F\varphi$, $F\psi$, & $\mathfrak{d}F\varphi$, $\mathfrak{d}F\psi$ seu per functiones, quæ harum differentialium factores sunt, exhibent, nempe:

1) $\frac{T(\psi \pm \varphi)}{ab} = \frac{T\psi . \mathfrak{S}\varphi . \dot{\mathfrak{S}}\varphi \pm T\varphi . \mathfrak{S}\psi . \dot{\mathfrak{S}}\psi}{(ab)^2 - T\varphi^2 . T\psi^2}$,

ubi $\mathfrak{S}\varphi \cdot \dot{\mathfrak{S}}\varphi = \mathfrak{d}T_\varphi$, & $\mathfrak{S}\varphi = \sqrt{a^2 + T\varphi^2}$, atque $\dot{\mathfrak{S}}_\varphi = \sqrt{b^2 + T_\varphi^2}$.

2) $\dfrac{S(\psi \pm \varphi)}{ac} = \dfrac{S\psi \cdot \dot{S}^c\varphi \cdot \mathfrak{T}^c\varphi \pm S\varphi \cdot \dot{S}^c\psi \cdot \mathfrak{T}^c\psi}{(ac)^2 - S\varphi^2 \cdot S\psi^2}$.

3) $\dfrac{\dot{S}(\psi \pm \varphi)}{bc} = \dfrac{\dot{S}\psi \cdot S^c\varphi \cdot \mathfrak{T}\varphi \pm \dot{S}\varphi \cdot S^c\psi \;\; \mathfrak{T}\psi}{(bc)^2 + \dot{S}\psi^2 \cdot \dot{S}\varphi^2}$.

4) $T^c(\psi \pm \varphi) = \dfrac{T^c\psi \cdot \dot{\mathfrak{S}}^c\varphi \cdot \mathfrak{S}^c\varphi \mp T^c\varphi \cdot \mathfrak{S}^c\psi \cdot \dot{\mathfrak{S}}^c\psi}{T^c\varphi^2 - T^c\psi^2} = \dfrac{T\varphi\,\mathfrak{d}T\psi \mp T\psi\,\mathfrak{d}T\psi}{T\varphi^2 - T\psi^2}$,

ubi $\mathfrak{S}^c\varphi \cdot \dot{\mathfrak{S}}^c\varphi = -\mathfrak{d}T^c\varphi$, & $\dot{\mathfrak{S}}^c\varphi = \sqrt{b^2 + T^c\varphi^2}$.

5) $S^c(\psi \pm \varphi) = c \cdot \dfrac{b^2 S^c\psi \cdot S^c\varphi \mp \dot{S}\varphi\,\mathfrak{T}\varphi \cdot \dot{S}\psi\,\mathfrak{T}\psi}{c^2\,\mathfrak{T}\varphi^2 - \dot{S}\varphi^2 \cdot S^c\psi^2}$, ubi $\dot{S}\varphi\,\mathfrak{T}\varphi = -\mathfrak{d}S^c\varphi$.

6) $\dot{S}^c(\psi \pm \varphi) = c \cdot \dfrac{a^2 \cdot \dot{S}^c\psi \cdot \dot{S}^c\varphi \mp S\varphi\,\mathfrak{T}^c\varphi \cdot S\psi\,\mathfrak{T}^c\psi}{c^2\,\widehat{\mathfrak{T}^c\varphi^2} + S\varphi^2 \cdot \dot{S}^c\psi^2}$; $\left\{\begin{array}{l}\text{ubi denominator}\\ = \widehat{ac}^2 - S\varphi^2 S\psi^2\end{array}\right\}$.

7) $\mathfrak{T}(\psi \pm \varphi) = b \cdot \dfrac{c^2\,\mathfrak{T}\varphi \cdot \mathfrak{T}\psi \pm \dot{S}\varphi\,S^c\varphi \cdot \dot{S}\psi\,S^c\psi}{b^2\,S^c\psi^2 + \mathfrak{T}\varphi^2 \cdot \dot{S}\psi^2}$, ubi

$\dot{S}\varphi \cdot S^c\varphi = \mathfrak{d}\mathfrak{T}\varphi$, & $b^2 \cdot S^c\psi^2 + \mathfrak{T}\varphi^2 \cdot \dot{S}\psi^2 = b^2 S^c\varphi^2 + \mathfrak{T}\psi^2 \cdot \dot{S}\varphi^2 = b^2 c^2 + S\varphi^2 S\psi^2$.

8) $\mathfrak{T}^c(\psi \pm \varphi) = a \cdot \dfrac{\dot{S}^c\psi^2 \cdot \mathfrak{T}^c\varphi^2 + b^2\,S\psi^2}{c^2\,\mathfrak{T}^c\varphi \cdot \mathfrak{T}^c\psi \pm S\varphi\,\dot{S}^c\varphi \cdot S\psi\,\dot{S}^c\psi}$.

$\mathfrak{T}^c(\psi \dot{\pm} \varphi) = a \cdot \dfrac{c^2\,\mathfrak{T}^c\varphi \cdot \mathfrak{T}^c\psi \mp S\varphi \cdot \dot{S}^c\varphi \cdot S\psi \cdot \dot{S}^c\psi}{(ac)^2 - S\varphi^2\,S\psi^2}$.

9) $\mathfrak{S}(\psi \pm \varphi) = a \cdot \dfrac{\mathfrak{S}\varphi^2 \cdot \dot{\mathfrak{S}}\psi^2 - c^2\,T\psi^2}{b^2 \cdot \mathfrak{S}\varphi\,\mathfrak{S}\psi \mp T\varphi\,\dot{\mathfrak{S}}\varphi \cdot T\psi\,\mathfrak{S}\psi}$, ubi numerator est $= a \cdot (b^2 \cdot (a^2 + T\varphi^2 + T\psi^2) + T\varphi^2 \cdot T\psi^2)$, ideoque ad φ & ψ Symmetricus, & $= a \cdot (\dot{\mathfrak{S}}\varphi^2 \cdot \dot{\mathfrak{S}}\psi^2 + b^2 c^2)$.

10) $\dot{\mathfrak{S}}(\psi \pm \varphi) = b \cdot \dfrac{\dot{\mathfrak{S}}\varphi^2\,\mathfrak{S}\psi^2 + c^2\,T\psi^2}{a^2 \cdot \dot{\mathfrak{S}}\varphi\,\dot{\mathfrak{S}}\psi \mp T\varphi\,\mathfrak{S}\varphi \cdot T\psi\,\mathfrak{S}\psi}$.

11) $\mathfrak{S}^c(\psi \pm \varphi) = \dfrac{\mathfrak{S}^c\psi^2 \cdot \mathfrak{S}^c\varphi^2 - \widehat{ac}^2}{\mathfrak{S}^c\varphi \cdot \dot{\mathfrak{S}}^c\psi\,T^c\psi \pm \mathfrak{S}^c\psi \cdot \dot{\mathfrak{S}}^c\varphi\,T^c\varphi}$.